

Speed of particles and a relativity of locality in κ -Minkowski quantum spacetime

Giovanni AMELINO-CAMELIA,^{1,2} Niccoló LORET,^{1,2} and Giacomo ROSATI^{1,2}

¹*Dipartimento di Fisica, Università di Roma "La Sapienza", P.le A. Moro 2, 00185 Roma, EU*

²*INFN, Sez. Roma1, P.le A. Moro 2, 00185 Roma, EU*

The last decade of research on κ -Minkowski noncommutative spacetime has been strongly characterized by a controversy concerning the speed of propagation of massless particles. Most arguments suggested that this speed should depend on the momentum of the particle strongly enough to be of interest for some ongoing experimental studies. But the only explicit derivations of worldlines in κ -Minkowski predicted no momentum dependence for the speed of massless particles. We return to this controversy equipped with the recent understanding (arXiv:1006.2126, arXiv:1007.0718, arXiv:1008.2962, arXiv:1101.0931) that in some quantum spacetimes coincidences of events assessed by an observer who is distant from the events can be artifactual, and particularly Smolin's thesis (arXiv:1007.0718) that κ -Minkowski should be an example of such a spacetime. We therefore set up our investigation in such a way that we never rely on the assessment of coincidences of events by distant observers. This allows us to verify explicitly that in κ -Minkowski simultaneously-emitted massless particles of different momentum are detected at different times, and establish a linear dependence of the detection times on momentum.

I. INTRODUCTION

Over the last decade there has been a strong effort aimed at seeking experimental evidence of Planck-scale features of the quantum-gravity and/or quantum-spacetime realm [1]. One of the most studied opportunities concerns the possibility that the speed of massless particles (photons) might have a Planck-scale-induced dependence on wavelength/momentum. On the theory side this finds motivation in several semi-heuristic but compelling analyses of quantum-gravity/quantum-spacetime frameworks which indeed expose plausible mechanisms for such a momentum dependence to arise [2–6]. And from the viewpoint of experimental tests it is noteworthy that some of these theory scenarios produce effects that are within the range of sensitivities of analyses exploiting data presently gathered with powerful gamma-ray telescopes [3, 7–10].

While the opportunity to look for effects motivated by work on some quantum-gravity/quantum-spacetime theories is certainly exciting, for the assessment of the possible impact of these studies on the development of quantum gravity it is crucial that we establish whether any models could be falsified. If we stumble upon positive/discovery experimental results it would evidently be a huge impulse for quantum-gravity research. But if it turns out that the results are negative (which of course is most likely) will we have learned anything substantial about the quantum-gravity problem? Would indeed some models be falsified?

From this perspective it remains of paramount importance to establish that the momentum dependence of the speed of massless particles is a definite prediction (rather than a plausible outcome) of some specific models of quantum gravity and/or quantum spacetime. The most promising opportunity of this type is found in studies of the κ -Minkowski quantum spacetime [6, 11–13], where several lines of analysis appear to be very close to a fully rigorous derivation of a result on the momentum dependence of the speed of massless particles. As a matter of fact we might already have such a robust result, the one reported in Ref. [6], exploiting the properties of the κ -Minkowski quantum differential calculus to derive constructively the speed of propagation of classical κ -Klein-Gordon waves. The main obstruction for adopting this momentum-dependence as a “consensus result” of the κ -Minkowski literature is found within an approach based on “ κ -Minkowski phase spaces”, where one can derive worldlines of massless particles, apparently finding no such momentum dependence [14].

Because of its relevance both for phenomenology and from a theory perspective this issue of the description of the speed of massless particles in κ -Minkowski has been hotly debated (see, *e.g.*, Refs. [6, 13–25]). We return to this issue equipped with the recent understanding [26–30] that in some quantum spacetimes coincidences of events assessed by an observer who is distant from the events can be artifactual, and particularly Smolin’s thesis [27] that κ -Minkowski should be an example of such a spacetime. We handle this “relativity of locality” by setting up our investigation of worldlines in κ -Minkowski in such a way that we never rely on the assessment of coincidences of events by distant observers. We find that in κ -Minkowski simultaneously-emitted massless particles of different momentum are detected at different times, and establish a linear dependence of the detection times on momentum, in full agreement with the previous independent derivation reported in Ref. [6].

II. TIME-TO-THE-RIGHT FORMULATION OF κ -MINKOWSKI

We shall focus throughout on the most studied formulation of theories in κ -Minkowski spacetime, often labeled as “bicrossproduct basis” [11, 13] or “time-to-the-right basis” [31].

We adopt the convention $\eta_{\mu\nu} = \{-1, 1, 1, 1\}$ for the Lorentzian metric, and we adopt units such that the speed-of-light scale (speed of massless particles in the infrared limit) and the reduced Planck constant are 1 ($c = 1 = \hbar$).

Starting from κ -Minkowski noncommutativity [11–13]

$$[x_j, x_0] = i\ell x_j, \quad (1)$$

in this formulation one introduces the Fourier transform $\tilde{\Phi}(k)$ of a given κ -Minkowski field $\Phi(x)$ using the time-to-

the-right convention

$$\Phi(x) = \int d^4x \tilde{\Phi}(k) e^{ik_j x^j} e^{ik_0 x^0} .$$

This is often equivalently described in terms of the time-to-the-right Weyl map \mathcal{W}_R by writing that

$$\Phi(x) = \mathcal{W}_R \left(\int d^4x \tilde{\Phi}(k) e^{ik_\mu x^\mu} \right)$$

where it is intended that coordinates trivially commute when placed inside the Weyl map, $\mathcal{W}_R(x_j x_0) = \mathcal{W}_R(x_0 x_j)$, and that taking a function out of the time-to-the-right Weyl map implies [31] time-to-the-right ordering, so that for example $x_j x_0 = \mathcal{W}_R(x_j x_0)$, $x_j x_0 = \mathcal{W}_R(x_0 x_j)$, and $e^{ik_j x^j} e^{ik_0 x^0} = \mathcal{W}_R(e^{ik_\mu x^\mu})$.

Then several arguments [11, 13], including the ones based on the recently-developed techniques of Noether analysis [32–34], lead one to find generators of symmetries under translations, space-rotations and boosts. For translations one has that

$$P_\mu e^{ik_j x^j} e^{ik_0 x^0} = k_\mu e^{ik_j x^j} e^{ik_0 x^0} = \mathcal{W}_R \left(-i \partial_\mu e^{ik_\nu x^\nu} \right) , \quad (2)$$

and in general $P_\mu \mathcal{W}_R(f(x)) = \mathcal{W}_R(-i \partial_\mu f(x))$.

Similarly one has that the generators of space rotations are given by

$$R_l e^{ik_j x^j} e^{ik_0 x^0} = \epsilon_{lmn} x_m k_n e^{ik_j x^j} e^{ik_0 x^0}$$

and the generators of boosts are given by

$$\mathcal{N}_l e^{ik_j x^j} e^{ik_0 x^0} = \left[x_0 k_l - x_l \left(\frac{1 - e^{-2\ell k_0}}{2\ell} + \frac{\ell}{2} k_m k^m \right) \right] e^{ik_j x^j} e^{ik_0 x^0}$$

The main properties of these generators are described, *e.g.*, in Refs. [11, 13, 31–34]. Here particularly relevant is the Casimir \mathcal{C} ,

$$\mathcal{C} = \left(\frac{2}{\ell} \right)^2 \sinh^2 \left(\frac{\ell}{2} P_0 \right) - e^{\ell P_0} P_j P^j .$$

Analyses based directly on this construction, such as the study [6] based on the Casimir and the properties of the κ -Minkowski quantum differential calculus [35, 36], lead to the conclusion that massless particles (or at least massless waves) propagate with momentum-dependent speed

$$v = e^{-\ell |\vec{p}|} \simeq 1 - \ell |\vec{p}| .$$

Before moving on to the next section, where we review the type of analysis that has been used to challenge this result, let us set up that discussion by noticing the following properties of the translation generators

$$[P_l, x_m] e^{ik_j x^j} e^{ik_0 x^0} = -i \delta_{lm} e^{ik_j x^j} e^{ik_0 x^0} ,$$

$$[P_0, x_0] e^{ik_j x^j} e^{ik_0 x^0} = i e^{ik_j x^j} e^{ik_0 x^0} .$$

$$[P_0, x_l] e^{ik_j x^j} e^{ik_0 x^0} = 0 ,$$

$$[P_l, x_0] e^{ik_j x^j} e^{ik_0 x^0} = -i \ell k_l e^{ik_j x^j} e^{ik_0 x^0} ,$$

where, for the last equation, we used the observation that $x_0 f(x) = f(x) x_0 + \ell x_j P^j f(x)$, which follows from (1)-(2).

III. κ -MINKOWSKI PHASE-SPACE CONSTRUCTION

Our next task is to review the “ κ -Minkowski phase-space construction” that has so far provided the only argument against the momentum dependence of the speed of massless particles in κ -Minkowski.

We work at first order in ℓ in a 1+1-dimensional κ -Minkowski spacetime. This simplified setup helps the clarity of presentation and contains faithfully the hotly-debated issue concerning the speed of massless particles in κ -Minkowski.

In the κ -Minkowski phase-space construction one describes classical worldlines of particles in terms of an auxiliary worldline parameter τ (we denote by \dot{Q} the τ derivative of an observable Q , so that $\dot{Q} \equiv \partial Q / \partial \tau$). The first ingredient of this derivation of worldlines is the following “ κ -Minkowski Poisson bracket” for the spacetime coordinates

$$\{x, t\} = -\ell x . \quad (3)$$

Then in light of the observations reported at the end of the previous section one describes the action of translation generators through the following Poisson brackets

$$\{\Omega, t\} = 1, \quad \{\Omega, x\} = 0 , \quad (4)$$

$$\{P, t\} = \ell P, \quad \{P, x\} = -1 . \quad (5)$$

And similarly one relies on the description of boosts here reviewed in the previous section in order to adopt the following Poisson brackets between generators of boosts and of translations:

$$\{\Omega, P\} = 0 , \quad \{\mathcal{N}, \Omega\} = P , \quad \{\mathcal{N}, P\} = \Omega + \ell \Omega^2 + \frac{\ell}{2} P^2 . \quad (6)$$

It is easy to check that (3),(4),(5),(6) satisfy all Jacobi identities. And from (6) one easily finds the κ -Minkowski deformed mass Casimir,

$$\mathcal{C} = \Omega^2 - P^2 + \ell \Omega P^2 . \quad (7)$$

One can then use [14] the mass Casimir as Hamiltonian of evolution of the observables on the worldline of a particle in terms of the worldline parameter τ . For this one starts by observing that Hamilton’s equations give the conservation of P and Ω along the worldlines

$$\dot{P} = \frac{\partial \mathcal{C}}{\partial x} = 0 , \quad \dot{\Omega} = -\frac{\partial \mathcal{C}}{\partial t} = 0 . \quad (8)$$

We shall denote with p and E these conserved values of P and Ω , and of course we denote with m^2 the conserved value of \mathcal{C} , so that in particular

$$m^2 = E^2 - p^2 + \ell E p^2 . \quad (9)$$

One then takes into account (5) in the derivation of the equations of motion:

$$\begin{aligned} \dot{t} &= \{\mathcal{C}, t\} = \frac{\partial \mathcal{C}}{\partial \Omega} \{\Omega, t\} + \frac{\partial \mathcal{C}}{\partial P} \{P, t\} = 2\Omega - \ell P^2 , \\ \dot{x} &= \{\mathcal{C}, x\} = \frac{\partial \mathcal{C}}{\partial \Omega} \{\Omega, x\} + \frac{\partial \mathcal{C}}{\partial P} \{P, x\} = 2P - 2\ell \Omega P . \end{aligned}$$

This evidently implies

$$\begin{aligned} t(\tau) &= t_0 + (2E - \ell p^2) \tau, \\ x(\tau) &= x_0 + (2p - 2\ell E p) \tau, \end{aligned}$$

from which, eliminating the parameter τ and imposing the Hamiltonian constraint $\mathcal{C} = m^2$, we find

$$x(p, x_0, t_0; t) = x_0 + \left(\frac{p}{\sqrt{p^2 + m^2}} - \ell p \left(1 - \frac{p^2}{p^2 + m^2} \right) \right) (t - t_0). \quad (10)$$

In particular, for massless particles these worldlines give a momentum-independent particle velocity:

$$x(p, x_0, t_0; t) = x_0 + t - t_0. \quad (11)$$

We shall not dwell much on the setup of this analysis. It suffices to notice that this is the only argument suggesting that massless particles propagate with undeformed (ℓ -independent) speed in κ -Minkowski. Some of us have elsewhere argued [19] that analyses such as the one of Ref. [6], which had found evidence of momentum dependence of the speed of massless particles, should have priority conceptually because they relied on the full quantum structure of the spacetime, including in particular the noncommutative differential calculus, which instead is moot in this derivation of worldlines. We can here ignore this debate since we shall find that when the analysis is advanced to the point of actually establishing physically-meaningful correlations between emission times, detection times and momentum, rather than merely a “coordinate velocity”, also the derivation of worldlines based on these “ κ -Minkowski phase-space constructions” produces results in full agreement with the differential-calculus-based analysis of Ref. [6].

IV. WHAT ABOUT BOB?

The result summarized in the previous section has been known for several years and was interpreted as a determination of the physical velocity of massless particles in κ -Minkowski. In classical spacetime with curvature it is well known that the coordinate velocity may be affected by coordinate artifacts (*e.g.* for an observer in classical de Sitter spacetime the speed of local photons is always 1, but this does not apply to the coordinate velocity that observer attributes to distant photons). Analogous coordinate artifacts for flat quantum spacetimes were neither expected nor found until very recently, with the first studies establishing the possibility of a “relative locality” in a quantum spacetime. This was encountered unexpectedly in the two independent studies¹ reported in Refs. [26, 27], and then cast into a more satisfactory and more ambitious conceptual framework in Ref. [29].

We shall here not need much of the broader picture² of the implications and possible formalizations of relative locality which was recently given in Ref. [29]. For our purposes here it is sufficient to appreciate the implications

¹ Before the first results on relativity of locality reported in Refs. [26, 27], there had been some papers arguing about the faith of locality in the relevant scenarios for spacetime quantization, but failing to contemplate the possibility that coincidences of events witnessed by local observers might not appear as such to observers distant from the events. In Ref. [37] it was observed that two particles on the same worldline for one observer might not be on the same worldline for another observer. Similarly in Ref. [38] it was observed that a particle could be inside a box for one observer but outside the box for another observer. And a reformulation of such a “box paradox” for locality was reported in Ref. [39], seeking a characterization that would be suitable for experimental testing. Ref. [41] argued that the violations of locality discussed in Ref. [39] could well be in no conflict with known experimental facts. Ref. [40] exposed the possibility of sizable violations of locality in computations involving time dilatation, and tentatively argued that there might be logically-consistent relativistic theories without an absolute objectivity of simultaneity at the same spatial point.

² We here intend to contribute to the analysis of the standard framework of κ -Minkowski phase spaces. In particular, consistently with the assumptions that are standard for work on κ -Minkowski phase spaces, we idealize the processes of emission and detection of particles. And we should also stress that, while one may legitimately argue [42] that some aspects of the structure of these κ -Minkowski phase-space constructions can be described in terms of the presence of some curvature in momentum space, the analysis of these κ -Minkowski phase-space constructions does not require one to refer directly to the geometry of momentum space. The formalism recently proposed in Ref. [29] is instead well suited for a description of emission and detection processes as controlled microscopic particle-physics processes, and hosts relative locality within a setup which places at center stage the geometry of momentum space.

of “relative locality” for coincidences of events, as described in a picture based on the introduction of spacetime coordinates, following the rather simple line of analysis advocated in Ref. [26]. Let us introduce this relative-locality characterization of coincident events by drawing a logical line through the history of relativistic theories. Starting with Galileian Relativity rest is relative (absolute rest requires a preferred frame, so it is not found in any relativistic theory), which in turn implies that the quality of being in the same (spatial) position at different times is not objective. But in Galileian Relativity simultaneity is still absolute: when one inertial observer establishes that two events have the same time coordinate then all other inertial observers agree. In Special Relativity distant simultaneity is observer dependent (“relative simultaneity” or “relative time”). The relativity of distant simultaneity in Special Relativity amounts to the fact that pairs of events which according to one inertial observer are simultaneous but occupy different spatial positions (events with some spatial distance between them), would not be simultaneous according to other inertial observers. But in Special Relativity it remains true that when two events coincide according to one observer they also coincide according to all other observers: when simultaneity concerns two events at the same spatial position it remains objective. This “absolute locality” of Special Relativity is in the residual objectivity of simultaneity, which is subject to the (spatial) distance between the events but is independent of the distance between the two events and the observer: locality is objective independently of whether it is a “distant locality” (coincidences of events far from the observer) or a “local locality” (coincidences of events that occur in the origin of the coordinate system of a given observer). It is now emerging that in at least certain quantum spacetimes [26, 27, 29] a weaker form of locality holds: events that are coincident and occur in the origin of a given observer Alice are objectively coincident for all observers that share Alice’s origin (all observers which are purely boosted with respect to Alice), but may not appear to be coincident to observers which are far from Alice. Locality is still objective, but the abstraction of “distant locality” is lost.

The possibility that κ -Minkowski might be one of these quantum spacetimes where the objectivity of distant locality is lost was already raised in Ref. [27]. We here fully establish that there is relative locality in κ -Minkowski, and we show that, when this is taken into account, also within the “ κ -Minkowski phase-space construction” one finds that simultaneously-emitted massless particles of different momentum are not detected simultaneously.

Amusingly a simple way to establish these facts can be based just on the worldlines rederived in the previous section. To see this it suffices to use the setup reviewed in the previous section to formalize a simultaneous emission occurring in the origin of an observer Alice. This will be described by Alice in terms of two worldlines, a massless particle with momentum p_1 and a massless particle with momentum p_2 , which actually coincide because of the momentum independence of the coordinate velocity:

$$x_{p_1}^A(t^A) = t^A, \quad (12)$$

$$x_{p_2}^A(t^A) = t^A. \quad (13)$$

It is useful to focus on the case of p_1 and p_2 such that $|p_1| \ll |p_2|$, and $|\ell p_1| \simeq 0$ (the particle with momentum p_1 is soft enough that it behaves as if $\ell = 0$) while $|\ell p_2| \neq 0$, in the sense that for the hard particle the effects of ℓ -deformation are not negligible.

A central role in our analysis is played by the translation transformations codified in (4),(5). These allow us to establish how the assignment of coordinates on points of a worldline differs between two observers connected by a generic translation \mathcal{T}_{a_t, a_x} , with component a_t along the t axis and component a_x along the x axis [14, 28]

$$\begin{aligned} x' &= x - a_t \{\Omega, x\} + a_x \{P, x\}, \\ t' &= t - a_t \{\Omega, t\} + a_x \{P, t\}. \end{aligned} \quad (14)$$

Using these we can look at the two Alice worldlines, (12) and (13), from the perspective of a second observer, Bob, at rest with respect to Alice at distance a from Alice (Bob = $\mathcal{T}_{a,a} \triangleright$ Alice), local to a detector that the two particles eventually reach. Of course, since we have seen that the coordinate velocity is momentum independent,

according to Alice's coordinates the two particles reach Bob simultaneously. But can this distant coincidence of events be trusted? The two events which according to the coordinates of distant observer Alice are coincident are the crossing of Bob's worldline with the worldline of the particle with momentum p_1 and the crossing of Bob's worldline with the worldline of the particle with momentum p_2 . To clarify the situation we should look at the two worldlines from the perspective of Bob, the observer who is local to the detection of the particles.

Evidently these Bob worldlines are obtained from Alice worldlines using the translation transformation codified in (4),(5). Acting on a generic Alice worldline $x^A(p^A, x_0^A, t_0^A; t^A)$ this gives a Bob worldline $x^B(p^B, x_0^B, t_0^B; t^B)$ as follows:

$$\begin{aligned} p^B &= p^A - a \{ \Omega, p^A \} + a \{ P, p^A \} = p^A , \\ x_0^B &= x_0^A - a \{ \Omega, x_0^A \} + a \{ P, x_0^A \} = x_0^A - a , \\ t_0^B &= t_0^A - a \{ \Omega, t_0^A \} + a \{ P, t_0^A \} = t_0^A - a + \ell a p . \end{aligned} \tag{15}$$

And specifically for the two worldlines of our interest, given for Alice in (12) and (13), one then finds

$$\begin{aligned} x_{p_1}^B(t^B) &= t^B - \ell a p_1 \simeq t^B , \\ x_{p_2}^B(t^B) &= t^B - \ell a p_2 . \end{aligned} \tag{16}$$

We have found that, because of the peculiarities of translational symmetries of the κ -Minkowski quantum spacetime, the two worldlines, which were coincident according to Alice, are distinct worldlines for Bob. According to Bob, who is at the detector, the two particles reach the detector at different times: $t^B \simeq 0$ for the soft particle and $t^B = \ell a p_2$ for the hard particle. And these are the two massless particles which, according to the observer Alice who is at the emitter, were emitted simultaneously.

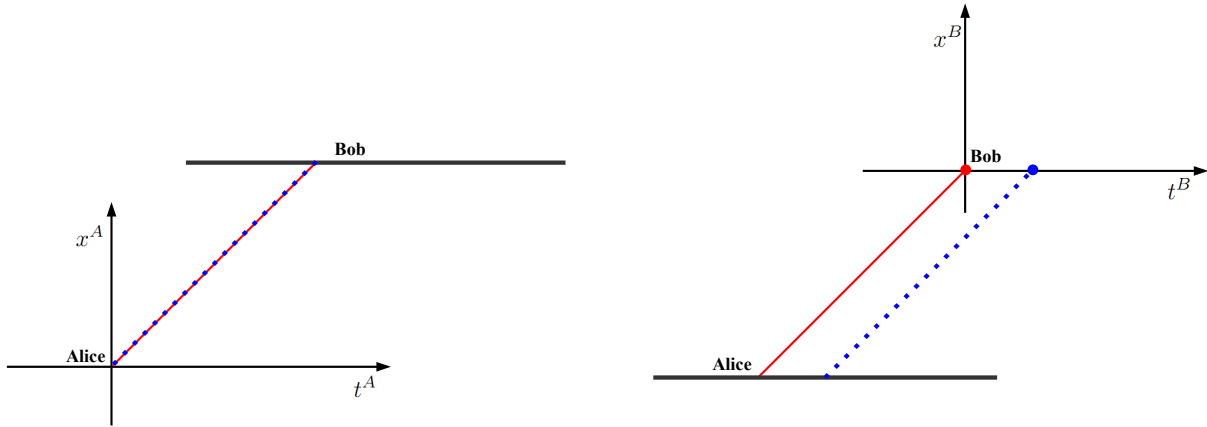


Figure 1. Two simultaneously-emitted massless particles of different momentum in κ -Minkowski are detected at different times. The figure shows how the simultaneous emission of two such particles and their non-simultaneous detection is described according to the coordinates of observer Alice (left panel), who is at the emitter, and according to the coordinates of observer Bob (right panel), who is at the detector.

Reassuringly this result for momentum-dependence of time of detection of simultaneously-emitted massless particles,

$$\Delta t = -\ell \Delta p ,$$

is in perfect agreement with the prediction previously established by a completely different argument in Ref. [6].

Considering the confused and long history of discussions of velocity in κ -Minkowski it is perhaps worth highlighting the soundness of the operative procedure by which we have determined this correlation between momentum of simultaneously-emitted particles and times of detection. Our procedure rests safely on the robust shoulders of the procedure for determining the physical velocity measured by inertial observers in classical Minkowski spacetime, and this connection is allowed by the fact that the properties of infrared massless particles (properties of massless particles in the infrared limit) are unaffected by the κ -Minkowski deformation. The distant synchronization of the clocks on emitter/Alice and on detector/Bob is evidently a special-relativistic synchronization, relying on exchanges of infrared massless particles. And we implicitly assumed that the relative rest of Alice and Bob is established by exchanges of infrared massless particles, so that indeed it can borrow from the special-relativistic operative definition of inertial observers in relative rest. By construction the distance a between Alice and Bob is also defined operatively just like in special relativity, with the only peculiarity that Alice and Bob should determine it by exchanging infrared massless particles. This setup guarantees that infrared massless particles are timed and observed in κ -Minkowski exactly as in classical Minkowski spacetime. The new element of the κ -Minkowski relativistic theory, concerning the “hard” (“high-momentum”) massless particles, then also acquires a sound operative definition by our procedure centered on the simultaneous emission (with simultaneity prudently established by the local observer/emitter Alice) of an infrared and a hard massless particle, then comparing the arrival times at observer/detector Bob (times of arrival prudently established according to the local observer, indeed Bob).

In the idealized setting of a sharply flat spacetime our procedure is applicable for any arbitrarily high value of the distance a . But of course for most realistic applications one will be interested in contexts where sharp flatness of spacetime cannot a priori be assumed, and evidently in such more general cases the limit $a \rightarrow 0$ of our procedure should be relied upon. Note however that the characterization of observations (local) and inferences (distant) given by the coordinates of Alice and Bob, which we summarized in Fig. 1, evidently remains valid even for small values of a : no matter how close Alice and Bob are, one still has that in Alice’s coordinates the detections at Bob appear to be simultaneous (while Bob, local to the detections, establishes that they are not simultaneous) and that in Bob’s coordinates the emission at Alice appears to be not simultaneous (while Alice, local to the emissions, establishes that they are simultaneous).

V. ASIDE ON CLASSICAL MINKOWSKI WITH NONCOMMUTATIVE COORDINATES

The robustness of our strategy of analysis extends even beyond what might be already expected with the observations we offered so far. Specifically, it is well suited for dealing even with the most virulent coordinate artifacts. In this section we provide further evidence of this robustness by applying our strategy of analysis to a very awkward description of the familiar classical Minkowski spacetime, a description that intentionally introduces severe coordinate artifacts.

We obtain such a description by using a momentum-dependent redefinition of spacetime coordinates recently introduced by Smolin in Ref. [27]:

$$\tilde{x} = x$$

$$\tilde{t} = t + \ell x P e^{-\ell \Omega} \simeq t + \ell x P$$

(where the last equality is valid for $|\ell\Omega| \ll 1$, which is the regime here of interest).

This redefinition of coordinates was proposed by Smolin as a way to probe certain properties of κ -Minkowski spacetime. We use the same redefinition in classical Minkowski spacetime, since it happens to introduce the type of severe coordinate artifacts that can serve our purposes here. Postponing some observations on the conceptual implications of this type of redefinition of coordinates to the next section, let us now proceed exposing the coordinate artifacts.

So we start by summarizing the properties of the 1+1D classical Minkowski spacetime, with its classical Poincaré symmetries:

$$\{\tilde{x}, \tilde{t}\} = 0 , \quad (17)$$

$$\{\Omega, \tilde{t}\} = 1, \quad \{\Omega, \tilde{x}\} = 0 , \quad (18)$$

$$\{P, \tilde{t}\} = 0, \quad \{P, \tilde{x}\} = -1 , \quad (19)$$

$$\{\Omega, P\} = 0 , \quad \{\mathcal{N}, \Omega\} = P , \quad \{\mathcal{N}, P\} = \Omega , \quad (20)$$

$$\mathcal{C} = \Omega^2 - \Pi^2 . \quad (21)$$

This is the standard setup which famously establishes the momentum independence of the speed of massless particles in classical Minkowski spacetime.

If we now perform the Smolin redefinition of coordinates, $\tilde{t} = t + \ell x P$, $\tilde{x} = x$, the result is a description of classical Minkowski spacetime in terms of κ -Minkowski coordinates:

$$\{x, t\} = -\ell x . \quad (22)$$

But then correspondingly the translations of classical Minkowski spacetime should have a deformed rule of action:

$$\{\Omega, t\} = \{\Omega, \tilde{t} - \ell \tilde{x} P\} = 1 , \quad \{\Omega, x\} = \{\Omega, \tilde{x}\} = 0 , \quad (23)$$

$$\{P, t\} = \{P, \tilde{t} - \ell \tilde{x} P\} = \ell P , \quad \{P, x\} = \{P, \tilde{x}\} = -1 . \quad (24)$$

We can use (22), (23), (24) and $\mathcal{C} = \Omega^2 - \Pi^2$ to derive the corresponding equations of motion:

$$\dot{t} = \{\mathcal{C}, t\} = \frac{\partial \mathcal{C}}{\partial \Omega} \{\Omega, t\} + \frac{\partial \mathcal{C}}{\partial P} \{P, t\} = 2\Omega - 2\ell P^2 , \quad (25)$$

$$\dot{x} = \{\mathcal{C}, x\} = \frac{\partial \mathcal{C}}{\partial \Omega} \{\Omega, x\} + \frac{\partial \mathcal{C}}{\partial P} \{P, x\} = 2P .$$

i.e.

$$t(\tau) = t_0 + (2E - 2\ell p^2) \tau , \quad (26)$$

$$x(\tau) = x_0 + 2p\tau .$$

Therefore the worldlines of particles in classical Minkowski spacetime, when described in terms of the coordinates x, t take the form:

$$x(p, x_0, t_0; t) = x_0 + \left(\frac{p}{\sqrt{p^2 + m^2}} + \ell p \frac{p^2}{p^2 + m^2} \right) (t - t_0) . \quad (27)$$

which for the massless case reduces to $x(p, x_0, t_0; t) = x_0 + (1 + \ell p)(t - t_0)$. So we have established that the adoption of the noncommutative coordinates x, t produces a coordinate velocity of massless particles in classical Minkowski spacetime which is momentum dependent. This must clearly be the result of severe coordinate artifacts introduced by our awkward adoption of noncommutative coordinates for classical Minkowski spacetime. So we have here an ideal opportunity to test the robustness of our strategy of analysis. Let us therefore consider a pair of massless particles of different momentum emitted simultaneously in the origin of observer Alice, an observer in classical Minkowski spacetime adopting the noncommutative coordinates x, t :

$$x_{p_1}^A = (1 + \ell p_1) t^A, \quad (28)$$

$$x_{p_2}^A = (1 + \ell p_2) t^A. \quad (29)$$

Following our strategy of analysis, we are interested in describing the detection of these two massless particles according to an observer Bob at the detector. A generic Alice worldline $x^A(p^A, x_0^A, t_0^A; t^A)$ is mapped into a Bob worldline $x^B(p^B, x_0^B, t_0^B; t^B)$ with

$$\begin{aligned} p^B &= p^A - a \{ \Omega, p^A \} + a \{ P, p^A \} = p^A, \\ x_0^B &= x_0^A - a \{ \Omega, x_0^A \} + a \{ P, x_0^A \} = x_0^A - a, \\ t_0^B &= t_0^A - a \{ \Omega, t_0^A \} + a \{ P, t_0^A \} = t_0^A - a + \ell a p. \end{aligned} \quad (30)$$

In particular Alice's worldlines (28) and (29) are mapped into Bob's

$$\begin{aligned} x_{p_1}^B(t^B) &= (1 + \ell p_1) t^B, \\ x_{p_2}^B(t^B) &= (1 + \ell p_2) t^B. \end{aligned}$$

So we see that both particles cross the origin of Bob. As of course expected (in spite of the awkward choice of coordinates) we have found that in classical Minkowski spacetime two massless particles with different momentum that were simultaneously emitted at Alice reach Bob simultaneously.

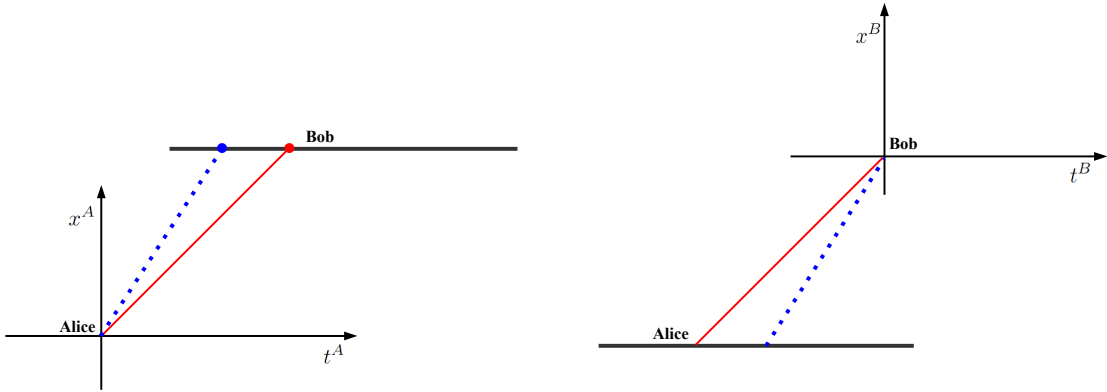


Figure 2. Two simultaneously-emitted massless particles of different momentum in classical Minkowski spacetime are detected simultaneously. The figure shows, assuming observers adopt awkward “ κ -Minkowski coordinates” for classical Minkowski spacetime, how the simultaneous emission of two such massless particles and their detection is described according to the coordinates of observer Alice (left panel), who is at the emitter, and according to the coordinates of observer Bob (right panel), who is at the detector.

VI. A FIRST LOOK AT OTHER FEATURES

In this section we comment briefly on some other issues that are relevant for the analysis of κ -Minkowski theories, for which our approach appears to be fruitful. A more in depth investigation of these topics is the subject of an ongoing investigation [43].

A. Momentum-dependent redefinitions of coordinates

From a perspective broader than the confines of κ -Minkowski studies perhaps the most striking aspect of our results resides in the observation that in presence of relativity of locality the equations of motion (the worldlines, in our case) written by an observer (say, Alice) are affected by coordinate artifacts. This also has implications for the idea of what may constitute “natural” redefinitions of coordinates, since it removes the main motivation for preferring to restrict one’s attention to redefinitions of coordinates that leave the equations of motion unchanged. Postponing a more detailed analysis of this point to the forthcoming Ref. [43] let us here offer some related comments focusing on the specific example of the change of coordinates for κ -Minkowski that was recently proposed by Smolin [27], who already for independent reasons contemplated the possibility of momentum-dependent redefinitions of spacetime coordinates.

We actually made use of Smolin’s redefinition of coordinates

$$\tilde{x} = x$$

$$\tilde{t} \simeq t + \ell x P$$

in the previous section, for an analysis of classical Minkowski spacetime. Smolin used [27] the same coordinate redefinition as a tool for probing the structure of theories in κ -Minkowski spacetime.

So one might ask when is this and/or a similar redefinition of coordinates appropriate and useful?

In relation to this question we should recall that evidently robust physical features of the relativistic theory are codified in the readout of clocks local to the emission of particles and (appropriately synchronized) clocks local to the detection of particles. This is something we ended up having to rely upon because the presence of relative locality spoiled the reliability (as codifiers of the physical content of the theory) of the equations of motion.

Momentum-dependent redefinition of spacetime coordinates, such as the one proposed by Smolin, evidently do not preserve the form of the equations of motion, but they can in some cases cause no harm to the readout of clocks local to the emission of particles and (appropriately synchronized) clocks local to the detection of particles. We see that Smolin’s redefinition of coordinates is an example of this: the redefinition is moot at $x = 0$ so it has no effect for the times an observer assigns to emission or detection events that she witnesses as local observer (in her origin with $x = 0$).

B. Generalized (Wigner-)Thomas rotations

Our next task is to characterize an aspect of the description of the symmetries of κ -Minkowski spacetime given in the previous sections in a way that provides a rather direct link to the relativity of locality and to the possibility of laws of momentum-dependent transformation of spacetime coordinates. An attempt to provide an in-depth description of this feature is ongoing [43], but we can here sketch out some preliminary observations.

Within ordinary special relativity it is well established that there is an intimate link between the relativity of simultaneity and (Wigner-)Thomas rotations [44–47]. We want to observe that there is an analogous link between

the type of relativity of locality found in our κ -Minkowski setup and a corresponding generalization of Thomas rotations.

Let us start within ordinary special relativity in a 3+1-dimensional spacetime (in the 1+1-dimensional case, whose simplicity we otherwise prefer, there are of course no Thomas rotations). For our purposes it suffices to consider two orthogonal boosts \mathcal{B}_1 and \mathcal{B}_2 . The composition of two such boosts is in general strongly characterized by the observation that (with $\mathcal{B}_j \simeq 1 + \xi_j \mathcal{N}_j$)

$$\mathcal{B}_2^{-1} \mathcal{B}_1^{-1} \mathcal{B}_2 \mathcal{B}_1 = 1 + \mathcal{B}_2^{-1} \mathcal{B}_1^{-1} \{ \mathcal{B}_2, \mathcal{B}_1 \} \simeq 1 - \xi_1 \xi_2 \{ N_2, N_1 \} . \quad (31)$$

In Galileian relativity one has that simultaneity is absolute and $\{N_1, N_2\} = 0$, which produces no Thomas rotation. In ordinary special relativity $\{N_1, N_2\} = -\epsilon_{123} R_3$, so that

$$\mathcal{B}_2^{-1} \mathcal{B}_1^{-1} \mathcal{B}_2 \mathcal{B}_1 = 1 - \xi_1 \xi_2 \{ N_2, N_1 \} = 1 - \xi_1 \xi_2 \epsilon_{123} R_3 . \quad (32)$$

It is well known that this Thomas rotation R_3 is a manifestation of length contraction: chaining boosts one ultimately produces rotations essentially because length contraction in one direction (and not in others) results into an overall rotation of axis.

Of course in ordinary special relativity no rotation of any sort is produced by chaining translations and boosts as follows (with $\mathcal{T} = 1 - a_t \Omega + a_x P$)

$$\mathcal{B}^{-1} \mathcal{T}^{-1} \mathcal{B} \mathcal{T} = 1 + \mathcal{B}^{-1} \mathcal{T}^{-1} \{ \mathcal{B}, \mathcal{T} \} \simeq 1 + a_t \xi \{ N, \Omega \} - a_x \xi \{ N, P \} , \quad (33)$$

focusing for simplicity on the case of a chain in which the spatial part of translations and the boosts are collinear. Indeed in ordinary special relativity $\{N_j, P_\alpha\}$ is a translation, $\{N_j, P_\alpha\} = \eta_{j\alpha} P_0 - \eta_{0\alpha} P_j$, and therefore (focusing again on the case of a chain in which the spatial part of translations and the boosts are collinear) one finds

$$\mathcal{B}^{-1} \mathcal{T}^{-1} \mathcal{B} \mathcal{T} = 1 + a_t \xi P - a_x \xi \Omega . \quad (34)$$

Postponing as mentioned a more detailed discussion of these issues to the forthcoming Ref. [43], let us comment briefly here on how these specific aspects of ordinary special relativity get modified in our κ -Minkowski-based framework with relativity of locality. First let us notice that in our (1+1-dimensional) construction we worked at all stages consistently with the possibility of generalizing all results to the case of 3+1 dimensions with classical Lorentz sector and classical space rotations, so we expect that also in κ -Minkowski $\mathcal{B}_2^{-1} \mathcal{B}_1^{-1} \mathcal{B}_2 \mathcal{B}_1 = 1 - \xi_1 \xi_2 \{ N_2, N_1 \} = 1 - \xi_1 \xi_2 \epsilon_{123} R_3$. In turn this leads us to expect that our κ -Minkowski-based framework hosts a rather ordinary mechanism of standard Thomas rotations, resulting from combinations of boosts.

The key novelty of our relativistic framework with relativity of locality is found combining boosts and translations. In fact, if we chain boosts and translations in κ -Minkowski we no longer get out a pure translation

$$\begin{aligned} \mathcal{B}^{-1} \mathcal{T}^{-1} \mathcal{B} \mathcal{T} &= \mathcal{B}^{-1} \mathcal{T}^{-1} \mathcal{T} \mathcal{B} + \mathcal{B}^{-1} \mathcal{T}^{-1} \{ \mathcal{B}, \mathcal{T} \} = 1 + \mathcal{B}^{-1} \mathcal{T}^{-1} \{ \mathcal{B}, \mathcal{T} \} \simeq \\ &\simeq 1 + \xi a_t \{ N, \Omega \} - \xi a_x \{ N, P \} = 1 + \xi a_t P - \xi a_x \left(\Omega + \ell \left(\Omega^2 + \frac{1}{2} P^2 \right) \right) \end{aligned}$$

And evidently the presence of the extra piece $\xi a_x \ell (\Omega^2 + P^2/2)$, which is an element of the universal enveloping algebra, ultimately produces a "Generalized (Wigner-)Thomas Rotation", here defined as a rotation in phase space³,

³ Within the confines of the κ -Minkowski framework one finds this explicit link from the properties of boosts and translations to "generalized Thomas rotations". It appears that in general, even setting aside the peculiarities of the κ -Minkowski framework, one should expect relative locality to produce modifications and generalizations of Thomas rotations and Thomas precessions. In particular, in Ref. [29] it was observed that one can expect, even before specifying anything about boosts, that a relative-locality-inducing nontrivial geometry of momentum space should provide opportunities for a novel mechanism of "Thomas precession in momentum space", a physical effect manifest in contexts where the evolution of a system is enclosing a loop in momentum space.

mixing spacetime and momentum-space coordinates, resulting from chaining boosts and translations. This is easily verified explicitly:

$$\mathcal{B}^{-1}\mathcal{T}^{-1}\mathcal{B}\mathcal{T} \triangleright x \simeq x + \xi a_t \{P, x\} - \xi a_x \left(\{\Omega, x\} + \ell \left(\{\Omega^2, x\} + \frac{1}{2} \{P^2, x\} \right) \right) \simeq x - \xi a_t + \ell \xi a_x p \quad (35)$$

$$\mathcal{B}^{-1}\mathcal{T}^{-1}\mathcal{B}\mathcal{T} \triangleright t \simeq t + \xi a_t \{P, t\} - \xi a_x \left(\{\Omega, t\} + \ell \left(\{\Omega^2, t\} + \frac{1}{2} \{P^2, t\} \right) \right) \simeq t - \xi a_x + \ell \xi a_t p - 2\ell \xi a_x E . \quad (36)$$

VII. CLOSING REMARKS

We have here settled a long-standing issue that strongly characterized the κ -Minkowski literature. In light of our results there is now full agreement between the two main techniques that had been competing for the description of the propagation of massless particles in κ -Minkowski. In fact, we found that, when the relativity of locality is appropriately taken into account, the analysis of worldlines within the κ -Minkowski-phase-space setup reproduces exactly the predictions previously obtained [6] with the alternative technique based mainly on the properties of the quantum differential calculus in κ -Minkowski.

As stressed in our opening remarks, this agreement among results obtained with different techniques of analysis of κ -Minkowski appears to be significant also for “quantum-gravity phenomenology” [1].

We also established here higher standards for the general objective of construction of relativistic theories with two non-trivial observer-independent scales, in the sense of the “Doubly Special Relativity” proposal [48–53]. Our set up based on the κ -Minkowski formalism evidently does host two such observer-independent scales: the speed-of-light scale (speed of photons in the infrared limit, not manifest in our formulas only because of conventions with $c = 1$) and the κ -Minkowski length/inverse-momentum scale ℓ . Particularly for the novel second observer-independent length scale the ability of mastering relativistic properties is still rather limited [54], but we here provided illustrative examples of several properties not previously contemplated.

We should also stress the role played by relative locality in the observations that led us to these results. Because of the nature of the context provided by κ -Minkowski phase-space constructions it turned out to be sufficient to rely on a rather rudimentary perspective on relative locality, which took shape in Refs. [26, 27], still centered on properties of spacetime coordinates. It would be interesting to verify whether the spacetime picture of these κ -Minkowski phase-space constructions could be found to emerge from some given choice of momentum-space geometry in the sense of the relative-locality framework proposed in Ref. [29].

ACKNOWLEDGEMENTS

We are very grateful to Michele Arzano, Francesco D’Andrea, Laurent Freidel, Giulia Gubitosi, Jerzy Kowalski-Glikman, A. Marcianó, Flavio Mercati, Lee Smolin and Anna Spinelli for comments on a first draft of this manuscript. The work of G. Amelino-Camelia was supported in part by grant RFP2-08-02 from The Foundational Questions Institute (fqxi.org).

[1] G. Amelino-Camelia, arXiv:0806.0339 [gr-qc].

[2] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, hep-th/9605211, Int. J. Mod. Phys. **A12**, 607 (1997).

[3] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos, D.V. Nanopoulos and S. Sarkar, astro-ph/9712103, Nature **393** (1998) 763.

- [4] R. Gambini and J. Pullin, Phys. Rev. **D59** (1999) 124021.
- [5] J. Alfaro, H.A. Morales-Tecotl and L.F. Urrutia, gr-qc/9909079, Phys. Rev. Lett. **84** (2000) 2318.
- [6] G. Amelino-Camelia and S. Majid, hep-th/9907110, Int. J. Mod. Phys. **A15** (2000) 4301.
- [7] A. Abdo *et al.*, Science **323** (2009) 1688.
- [8] John Ellis, N.E. Mavromatos, D.V. Nanopoulos, arXiv:0901.4052, Phys. Lett. **B674** (2009) 83.
- [9] G. Amelino-Camelia and L. Smolin, arXiv:0906.3731, Phys. Rev. **D80** (2009) 084017
- [10] A. A. Abdo et al, Nature **462** (2009) 331.
- [11] S. Majid and H. Ruegg, Phys. Lett. **B334** (1994) 348.
- [12] S. Zakrzewski, J. Phys. **A27** (1994) 2075-2082.
- [13] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. **243** (1995) 90.
- [14] M. Daszkiewicz, K. Imilkowska and J. Kowalski-Glikman, hep-th/0304027, Phys. Lett. **A323** (2004) 345.
- [15] J. Kowalski-Glikman, hep-th/0107054, Mod.Phys.Lett. **A17** (2002) 1
- [16] T. Tamaki, T. Harada, U. Miyamoto and T. Torii, gr-qc/0111056, Phys.Rev. **D65** (2002) 083003.
- [17] J. Lukierski and A. Nowicki, hep-th/0207022, Acta Phys.Polon. **B33** (2002) 2537.
- [18] P. Kosinski and P. Maslanka, hep-th/0211057, Phys. Rev. **D68** (2003) 067702.
- [19] G. Amelino-Camelia, F. D'Andrea and G. Mandanici, hep-th/0211022, JCAP **0309** (2003) 006
- [20] S. Mignemi, hep-th/0302065, Phys. Lett. **A316** (2003) 173.
- [21] K.-I. Tezuka, hep-th/0302126
- [22] R. Aloisio, A. Galante, A.F. Grillo, E. Luzio and F. Mendez, gr-qc/0410020, Phys. Rev. **D70** (2004) 125012.
- [23] S. Majid, hep-th/0604130.
- [24] S. Ghosh, hep-th/0608206, Phys. Rev. **D74** (2006) 084019
- [25] S. Ghosh and P. Pal, hep-th/0702159, Phys. Rev. **D75** (2007) 105021.
- [26] G. Amelino-Camelia, M. Matassa, F. Mercati and G. Rosati, arXiv:1006.2126, Phys. Rev. Lett. **106**, 071301 (2011).
- [27] L. Smolin, arXiv:1007.0718.
- [28] M. Arzano, J. Kowalski-Glikman, arXiv:1008.2962.
- [29] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman L. Smolin, *The principle of relative locality*, arXiv:1101.0931
- [30] L. Freidel, *The geometry of momentum space*, in preparation.
- [31] A. Agostini, G. Amelino-Camelia, F. D'Andrea, hep-th/0306013, Int. J.Mod. Phys. **A19** (2004) 5187
- [32] A. Agostini, G. Amelino-Camelia, M. Arzano, A. Marcianò and R. A. Tacchi, hep-th/0607221, Mod. Phys. Lett. **A22** (2007) 1779.
- [33] L. Freidel, J. Kowalski-Glikman and S. Nowak, arXiv:0706.3658, Int. J. Mod. Phys. **A23** (2008) 2687.
- [34] G. Amelino-Camelia, G. Gubitosi, A. Marciano, P. Martinetti and F. Mercati, arXiv:0707.1863, Phys. Lett. **B671** (2009) 298.
- [35] A. Sitarz, Phys. Lett. **B349** (1995) 42
- [36] S. Majid and R. Oeckl Commun. Math. Phys. **205** (1999) 617
- [37] G. Amelino-Camelia, gr-qc/0210063, Int. J. Mod. Phys. **D11** (2002) 1643
- [38] R. Schutzhold and W.G. Unruh, gr-qc/0308049, JETP Lett. **78** (2003) 431
- [39] S. Hossenfelder, Phys. Rev. Lett. **104** (2010) 140402.
- [40] U. Jacob, F. Mercati, G. Amelino-Camelia and T. Piran, arXiv:1004.0575, Phys. Rev. D **82** (2010) 084021
- [41] L. Smolin, arXiv:1004.0664
- [42] J. Kowalski-Glikman, hep-th/0207279, Phys. Lett. **B547** (2002) 291.
- [43] G. Amelino-Camelia, G. Gubitosi, N. Loret, F. Mercati, G. Rosati and A. Spinelli, in preparation.
- [44] L.H. Thomas, Nature **117** (1926) 514
- [45] E.P. Wigner, Ann. Math. **40** (1939) 149
- [46] R. Ferraro and M. Thibault, Eur. J. Phys. **20** (1999) 143
- [47] K. O'Donnell and M. Visser, arXiv:1102.2001
- [48] G. Amelino-Camelia, gr-qc/0012051, Int. J. Mod. Phys. **D11** (2002) 35; hep-th/0012238, Phys. Lett. **B510** (2001) 255.
- [49] J. Kowalski-Glikman, hep-th/0102098, Phys. Lett. **A286** (2001) 391.
- [50] J. Magueijo, L. Smolin, hep-th/0112090, Phys. Rev. Lett. **88** (2002) 190403.
- [51] G. Amelino-Camelia, gr-qc/0207049. Nature **418** (2002) 34
- [52] J. Magueijo, L. Smolin, gr-qc/0207085, Phys. Rev. **D67** (2003) 044017.
- [53] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. **D12** (2003) 299
- [54] G. Amelino-Camelia, arXiv:1003.3942, Symmetry **2** (2010) 230